Solving Triangles

The Shadow Problem

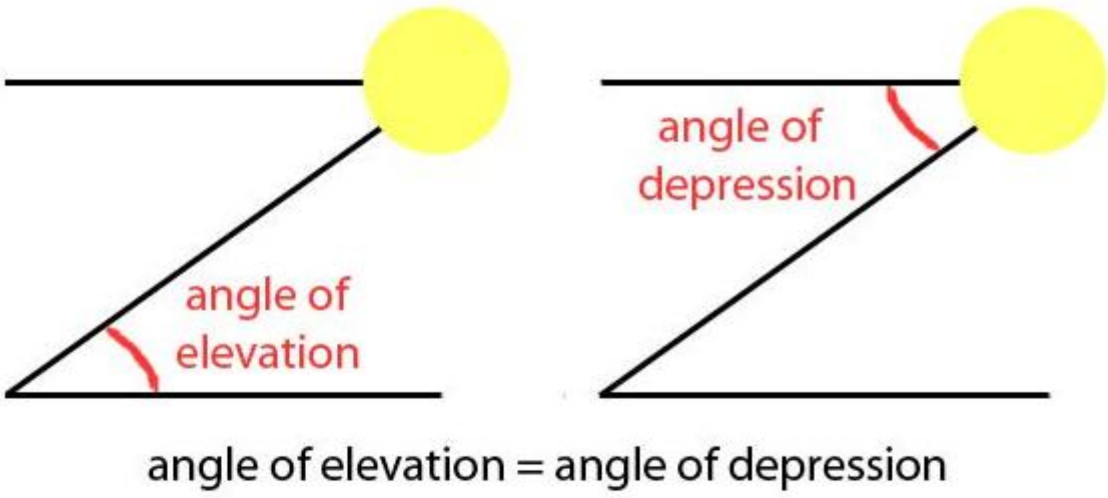
In the 1991 film Shadows and Fog, the eerie shadow of a larger-than-life figure appears against the wall as the shady figure lurks around the corner. How tall is the ominous character really? Filmmakers use the geometry of shadows and triangles to make this special effect.

The shadow problem is a standard type of problem for teaching trigonometry and the geometry of triangles. In the standard shadow problem, several elements of a triangle will be given. The process by which the rest of the elements are found is referred to as solving a triangle.

Basic Description

A triangle has six total elements: three sides and three angles. Sides are valued by length, and angles are valued by degree or radian measure. According to postulates for congruent triangles, given three elements, other elements can always be determined as long as at least one side length is given. Math problems that involve solving triangles, like shadow problems, typically provide certain information about just a few of the elements of a triangle, so that a variety of methods can be used to solve the triangle.

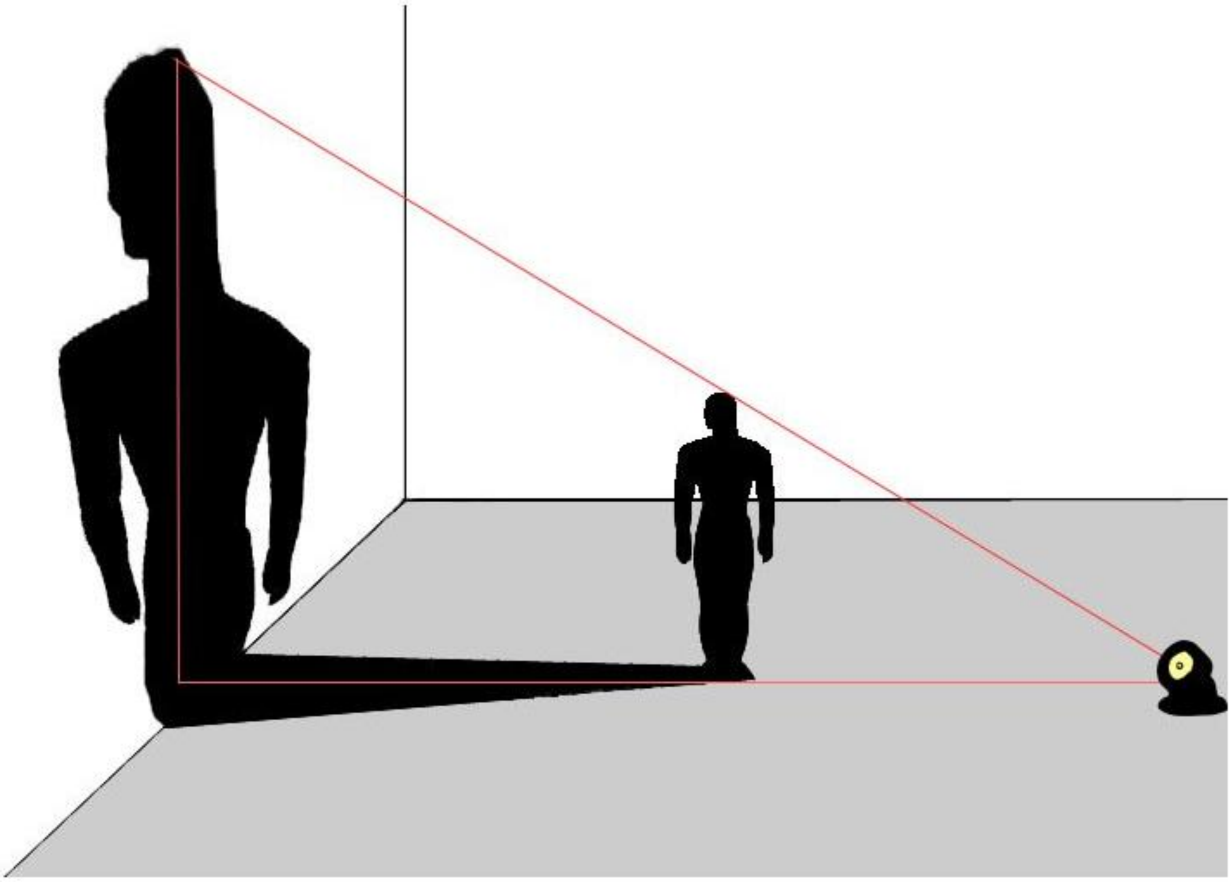
Shadow problems normally have a particular format. Some light source, often the sun, shines down at a given angle of elevation. The angle of elevation is the smallest—always acute—numerical angle measure that can be measured by swinging from the horizon from which the light source shines. Assuming that the horizon is parallel to the surface on which the light is shining, the angle of elevation is always equal to the angle of depression. The angle of depression is the angle at which the light shines down, compared to the angle of elevation which is the angle at which someone or something must look up to see the light source. Knowing the angle of elevation or depression can be helpful because trigonometry can be used to relate angle and side lengths.



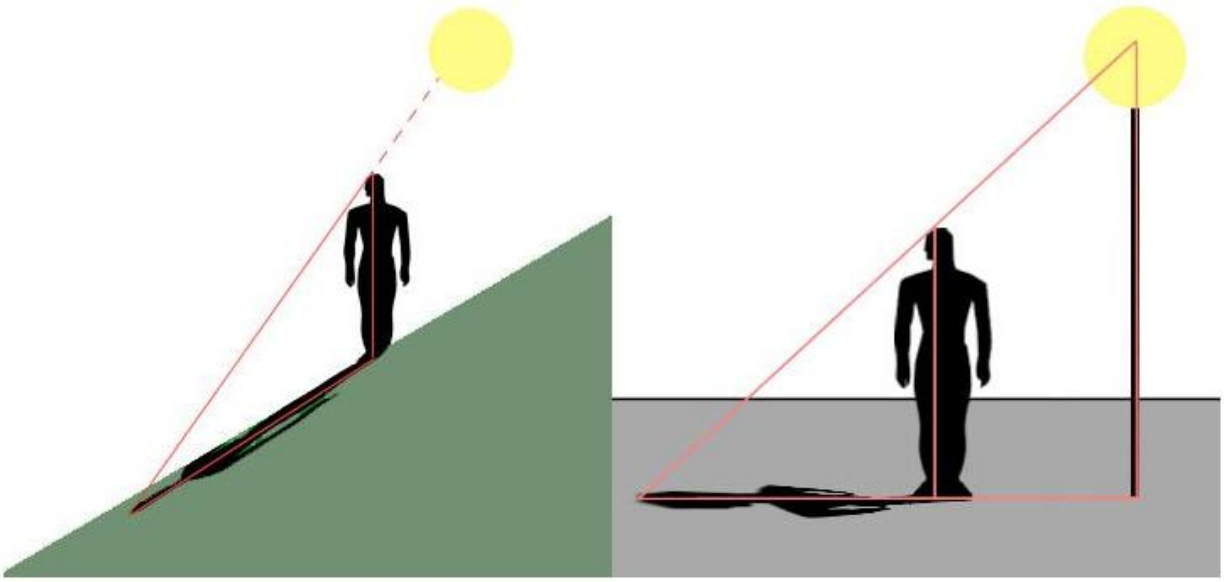
In the typical shadow problem, the light shines down on an object or person of a given height. It casts a shadow on the ground below, so that the farthest tip of the shadow makes a direct line with the tallest point of the person or object and the light source. The line that directly connects the tip of the shadow and the tallest point of the object that casts the shadow can be viewed as the hypotenuse of a triangle. The length from the tip of the shadow to the point on the surface where the object stands can be viewed as the first leg, or base, of the triangle, and the height of the object can be viewed as the second leg of the triangle. In the most simple shadow problems, the triangle is a right triangle because the object stands perpendicular to the ground.

In the picture below, the sun casts a shadow on the man. The length of the shadow is the base of the triangle, the height of the man is the height of the triangle, and the length from the tip of the shadow to top of the man's head is the hypotenuse. The resulting triangle is a right triangle.

In another version of the shadow problem, the light source shines from the same surface on which the object or person stands. In this case the shadow is projected onto some wall or vertical surface, which is typically perpendicular to the first surface. In this situation, the line that connects the light source, the top of the object and the tip of the shadow on the wall is the hypotenuse. The height of the triangle is the length of the shadow on the wall, and the distance from the light source to the base of the wall can be viewed as the other leg other leg of the triangle. The picture below diagrams this type of shadow problem, and this page's main picture is an example of one of these types of shadows.



More difficult shadow problems will often involve a surface that is not level, like a hill. The person standing on the hill does not stand perpendicular to the surface of the ground, so the resulting triangle is not a right triangle. Other shadow problems may fix the light source, like a street lamp, at a given height. This scenario creates a set of two similar triangles.

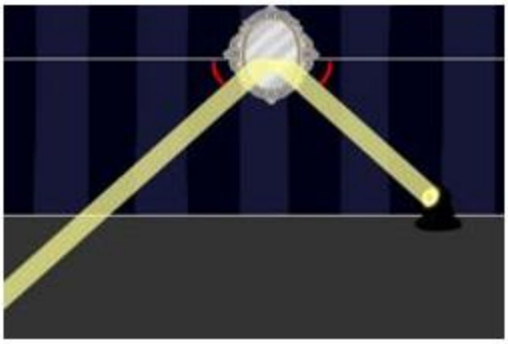


Ultimately, a shadow problem asks you to solve a triangle given only a few elements of the possible six total. In the case of some shadow problems, like the one that involves two similar triangles, information about one triangle may be given and the question may ask to find elements of another.

Why Shadows?

Shadows are useful in the set-up of a triangle problems because of the way light works. A shadow is cast when light cannot shine through a solid surface. Light shines in a linear fashion, that is to say it does not bend. Light waves travel forward in the same direction in which the light was shined.

In addition to the linear fashion in which light shines, light has certain angular properties. When light shines on an object that reflects light, it reflects back at the same angle at which it shined. Say a light shines onto a mirror. The angle between the beam of light and the wall that the mirror is the angle of approach. The angle from the wall at which the light reflects off of the mirror is the angle of departure. The angle of approach is equal to the angle of departure.



Light behaves the same way a cue ball does when it is bounced off of the wall of a pool table at a certain angle. Just like the way that the cue ball bounces off the wall, light reflects off of the mirror at exactly the same angle at which it shines. The beam of light has the same properties as the cue ball in this case: the angle of departure is the same as the angle of approach. This property will help with certain types of triangle problems, particularly those that involve mirrors.

More Than Just Shadows

Shadow problems are just one type of problem that involves solving triangles. There are numerous other formats and set ups for unsolved triangle problems. Most of these problems are formatted as word problems; they set up a triangle problem in terms of some real life scenario.

There are, however, many problems that simply provide numbers that represent angles and side lengths. In this type of problem, angles are denoted with capital letters, {A, B, C,...}, and the sides are denoted by lower-case letters, {a,b,c,...}, where a is the side opposite the angle A.

Ladder Problems

One other common problem in solving triangles is the ladder problem. A ladder of a given length is leaned up against a wall that stands perpendicular to the ground. The ladder can be adjusted so that the top of the ladder sits higher or lower on the wall and the angle that the ladder makes with the ground increases or decreases accordingly. Because the ground and the wall are perpendicular to one another, the triangles that need to be solved in ladder problems always have right angles. Since the right angle is always fixed, many ladder problems require the angle between the ground and the ladder, or the angle of elevation, to to be somehow associated with a fixed length of a ladder and the height of the ladder on the wall. In other words, ladder problems normally ask for the height of the ladder on the wall or the ground distance between the ladder and the wall, and typically require some trigonometric calculation.

Mirror Problems

Mirror problems are a specific type of triangle problem which involves two people or objects that stand looking into the same mirror. Because of the way a mirror works, light reflects back at the same angle at which it shines in, as explained below in A More Mathematical Explanation. In a mirror problem, the angle at which one person looks into the mirror, or the angle of vision is the same exact angle at which the second person must look into the mirror to make eye contact. Typically, the angle at which one person looks into the mirror is given along with some other piece of information. Once that angle is known, then one angle of the triangle is automatically known since the light reflects back off of the mirror at the same angle, making the angle of the triangle next to the mirror the supplement to twice the angle of vision.

Sight Problems

Like shadow problems, sight problems include many different scenarios and several forms of triangles. Most sight problems are set up as word problems. They involve a person standing below or above some other person or object. In most of these problems, a person measures an angle with a tool called an astrolabe or a protractor.

In the most standard type of problem, a person uses the astrolabe to measure the angle at which he looks up or down at something. In the example at the right, the bear stands in a tower of a given height and uses the astrolabe to measure the angle at which he looks down at the forest fire. The problem asks to find how far away the forest fire is from the base of the tower given the previous information.

Ways to Solve Triangles

In all cases, a triangle problem will only give a few elements of a triangle and will ask to find one or more of the lengths or angle measures that is not given. There are numerous formulas, methods, and operations that can help to solve a triangle depending on the information given in the problem.

The first step in any triangle problem is drawing a diagram. A picture can help to show which elements of the triangle are given and which elements are adjacent or opposite one another. By knowing where the elements are in relation to one another, we can use the trigonometric functions to relate angle and side lengths.

There are many techniques which can be implemented in solving triangles:

- **Trigonometry:** The basic trigonometric functions relate side lengths to angles. By substituting the appropriate values into the formulas for sine, cosine, or tangent, trigonometry can help to solve for a particular side length or angle measure of a right triangle. This is useful when given a side length and an angle measure.

- **Inverse Trigonometry:** Provided two side lengths, the inverse trig functions use the ratio of the two lengths and output an angle measure in right triangle trigonometry. Inverse trig is particularly useful in finding an angle measure when two side lengths are given in a right triangle.
- **Special Right Triangles:** Special right triangles are right triangles whose side lengths produce a particular ratio in trigonometry. A $30°-60°-90°$ triangle has a hypotenuse that is twice as long as one of its legs. A $45°-45°-90°$ is called an isosceles right triangle since both of its legs are the same length. These special cases can help to quicken the process of solving triangles.
- **Pythagorean Theorem:** The Pythagorean Theorem relates the squares of all three side lengths to one another in right triangles. This is useful when a triangle problem provides two side lengths and a third is needed.
  $a^{2}+b^{2} = c^{2}$
- **Pythagorean Triples:** A Pythagorean triple is a set of three positive integers that satisfy the Pythagorean Theorem. The set {3,4,5} is one of the most commonly seen triples. Given a right triangle with legs of length 3 and 4, for example, the hypotenuse is known to be 5 by Pythagorean triples.
- **Law of Cosines:** The law of cosines is a generalization of the Pythagorean Theorem which can be used for solving non-right triangles. The law of cosines relates the squares of the side lengths to the cosine of one of the angle measures. This is particularly useful given a SAS configuration, or when three side lengths are known and no angles for non-right triangles.
  $c^{2} = a^{2} + b^{2} - 2ab \cos C$
- **Law of Sines:** The law of sines is a formula that relates the sine of a given angle to the length of its opposite side. The law of sines is useful in any configuration when an angle measure and the length of its opposite side are given. It is also useful given an ASA configuration, and often the ASS configuration . The ASS configuration is known as the ambiguous case since it does not always provide one definite solution to the triangle.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

When solving a triangle, one side length must always be given in the problem. Given an AAA configuration, there is no way to prove congruency. According to postulates for congruent triangles, the AAA configuration proves similarity in triangles, but there is no way to find the side lengths of a triangle. Knowing just angle measures is not helpful in solving triangles.

Example Triangle Problems

Example 1: Using Trigonometry

A damsel in distress awaits her rescue from the tallest tower of the castle. A brave knight is on the way. He can see the castle in the distance and starts to plan his rescue, but he needs to know the height of the tower so he can plan properly. The knight sits on his horse 500 feet away from the castle. He uses his handy protractor to find the measure of the angle at which he looks up to see the princess in the tower, which is 15°. Sitting on the horse, the knight's eye level is 8 feet above the ground. What is the height of the tower?

Answer

We can use tangent to solve this problem. For a more in depth look at tangent, see Basic Trigonometric Functions.

Use the definition of tangent.

$$\tan =\frac{\text{opposite}}{\text{adjacent}}$$

Plug in the angle and the known side length.

$$\tan 15^\circ =\frac{x \text{ft}}{500 \text{ft}}$$

Clearing the fraction gives us

$$\tan 15^\circ (500) =x$$

Simplify for

$$(.26795)(500) =x$$

Round to get

$$134 \text{ft} \approx x$$

But this is only the height of the triangle and not the height of the tower. We need to add 8 ft to account for the height between the ground and the knight's eye-level which served as the base of the triangle.

$$134 \text{ft} + 8 \text{ft} = h$$

simplifying gives us

$$142 \text{ft} = h$$

The tower is approximately 142 feet tall.

Example 2: Using Law of Sines

A man stands 100 feet above the sea on top of a cliff. The captain of a white-sailed ship looks up at a 45° angle to see the man, and the captain of a black-sailed ship looks up at a 30° angle to see him. How far apart are the two ships?

Answer

To solve this problem, we can use the law of sines to solve for the bases of the two triangles since we have an AAS configuration with a known right angle. To find the distance between the two ships, we can take the difference in length between the bases of the two triangles.

First, we need to find the third angle for both of the triangles. Then we can use the law of sines.

For the black-sailed ship,

$$180^\circ - 90^\circ - 30^\circ =60^\circ$$

Let the distance between this ship and the cliff be denoted by b.

By the law of sines,

$$\frac{100}{\sin 30^\circ} = \frac{b}{\sin 60^\circ}$$

Clear the fractions to get,

$$100(\sin 60^\circ) = b(\sin 30^\circ)$$

Compute the sines of the angle to give us

$$100\frac{\sqrt{3}}{2} = b\frac{1}{2}$$

Simplify for

$$100(\sqrt{3}) = b$$

Simplify for

$a = 100 \text{ft}$

Multiply and round for

$b = 173 \text{ft}$

The distance between the two ships, x, is the positive difference between the lengths of the bases of the triangle.

$b-a=x$

$173-100 = 73 \text{ft}$

The ships are about 73 feet apart from one another.

For the white-sailed ship,

$180^\circ - 90^\circ - 45^\circ = 45^\circ$

Let the distance between this ship and the cliff be denoted by a.

By the law of sines,

$\frac{100}{\sin 45^\circ} = \frac{a}{\sin 45^\circ}$

Clear the fractions to get,

$100(\sin 45^\circ) = b(\sin 45^\circ)$

Compute the sines of the angle to give us

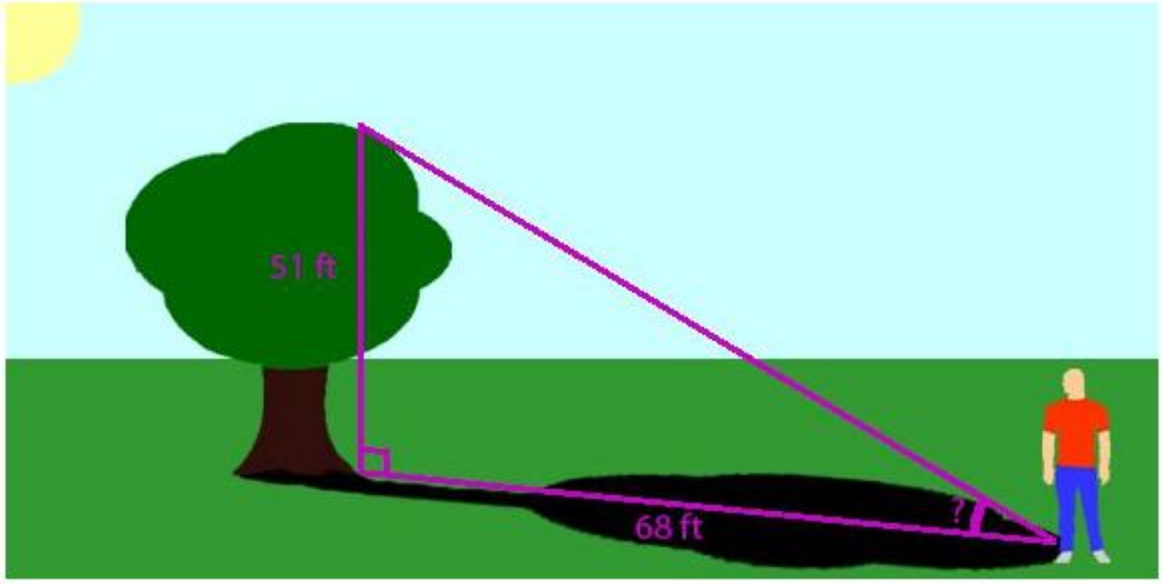
$100\frac{\sqrt{2}}{2} = b\frac{\sqrt{2}}{2}$

Simplify for

$a = 100 \text{ft}$

Example 3: Using Multiple Methods

At the park one afternoon, a tree casts a shadow on the lawn. A man stands at the edge of the shadow and wants to know the angle at which the sun shines down on the tree. If the tree is 51 feet tall and if he stands 68 feet away from the tree, what is the angle of elevation?



Answer

There are several ways to solve this problem. The following solution uses a combination of the methods described above.

First, we can use Pythagorean Theorem to find the length of the hypotenuse of the triangle, from the tip of the shadow to the top of the tree.

$a^{2}+b^{2} = c^{2}$

Substitute the length of the legs of the triangle for a, b

$51^{2}+68^{2} = c^{2}$

Simplifying gives us

$2601+4624 = c^{2}$

$7225 = c^{2}$

Take the square root of both sides for

$\sqrt{7225} = c$

$85 = c$

Next, we can use the law of cosines to find the measure of the angle of elevation.

$a^{2}=b^{2}+c^{2} - 2bc \cos A$

Plugging in the appropriate values gives us

$51^{2}=68^{2}+85^{2} - 2(85)(68) \cos A$

Computing the squares gives us

$2601= 4624+7225 - 11560 \cos A$

Simplify for

$2601= 11849 - 11560 \cos A$

Subtract 11849 from both sides for

$-9248 = -11560 \cos A$

Simplify to get

$.8 = \cos A$

Use inverse trigonometry to find the angle of elevation.

$A = 37^\circ$

Why It's Interesting

Shadow Problems are one of the most common types of problems used in teaching trigonometry. A shadow problem sets up a scenario that is simple, visual, and easy to remember. Shadow problems are commonly used and highly applicable.

Shadows, while an effective paradigm in a word problem, can even be useful in real life applications. In this section, we can use real life examples of using shadows and triangles to calculate heights and distances.

Example: Sizing Up Swarthmore

The Clothier Bell Tower is the tallest building on Swarthmore College's campus, yet few people know exactly how tall the tower stands. We can use shadows to determine the height of the tower. Here's how:

Step 1) Mark the shadow of the of the tower. Make sure to mark the time of day. The sun is at different heights throughout the day. The shadows are longest earlier in the morning and later in the afternoon. At around midday, the shadows aren't very long, so it might be harder to find a good shadow. When we marked the shadow of the bell tower, it was around 3:40 pm in mid-June.

Step 2) After marking the shadow, we can measure the distance from our mark to the bottom of the tower. This length will serve as the base of our triangle. In this case, the length of the shadow was 111 feet.

Step 3) Measure the angle of the sun at that time of day.

Use a yardstick to make a smaller, more manageable triangle. Because the sun shines down at the same angle as it does on the bell tower, the small triangle and the bell tower's triangle are similar and therefore have the same trigonometric ratios.

- Stand the yardstick so it's perpendicular to the ground so that it forms a right angle. The sun will cast a shadow. Mark the end of the shadow with a piece of chalk.
- Measure the length of the shadow. This will be considered the length of the base of the triangle.

Draw a diagram of the triangle made by connecting the top of the yardstick to the marked tip of the shadow.

Use inverse trigonometry to determine the angle of elevation.

$$\tan X = \frac{36 \text{in}}{27 \text{in}}$$

$$\arctan \frac{36}{27} = X$$

$$\arctan \frac{4}{3} = X$$

$$X = 53^\circ$$

Step 4)

Now, we can use trigonometry to solve the triangle for the height of the bell tower.

$$\tan 53^\circ = \frac{h}{111 \text{ft}}$$

Clearing the fractions,

$$111 (\tan 53^\circ) = h$$

Plugging in the value of $\tan 53^\circ$ gives us

$$111 \frac{4}{3} = h$$

Simplify for

$$148 \text{ft} = h$$

According to our calculations, the height of the Clothier Bell Tower is 148 feet.

History: Eratosthenes and the Earth

In ancient Greece, mathematician Eratosthenes made a name for himself in the history books by calculating the circumference of the Earth by using shadows. Many other mathematicians had attempted the problem before, but Eratosthenes was the first one to actually have any success. His rate of error was less than 2%.

Eratosthenes used shadows to calculate the distance around the Earth. As an astronomer, he determined the time of the summer solstice when the sun would be directly over the town of Syene in Egypt (now Aswan). On this day, with the sun directly above, there were no shadows, but in Alexandria, which is about 500 miles north of Syene, Eratosthenes saw shadows. He calculated based on the length of the shadow that the angle at which the sun hit the Earth was 7°. He used this calculation, along with his knowledge of geometry, to determine the circumference of the Earth.